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A model for large-scale structures in turbulent shear flows

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A procedure based on energy stability arguments is presented as a method for extracting large-scale, coherent structures from fully turbulent shear flows. By means of two distinct averaging operators, the instantaneous flow field is decomposed into three components: a spatial mean, coherent field and random background fluctuations. The evolution equations for the coherent velocity, derived from the Navier–Stokes equations, are examined to determine the mode that maximizes the growth rate of volume-averaged coherent kinetic energy. Using a simple closure scheme to model the effects of the background turbulence, we find that the spatial form of the maximum energy growth modes compares well with the shape of the empirical eigenfunctions given by the proper orthogonal decomposition. The discrepancy between the eigenspectrum of the stability problem and the empirical eigenspectrum is explained by examining the role of the mean velocity field. A simple dynamic model which captures the energy exchange mechanisms between the different scales of motion is proposed. Analysis of this model shows that the modes which attain the maximum amplitude of coherent energy density in the model correspond to the empirical modes which possess the largest percentage of turbulent kinetic energy. The proposed method provides a means for extracting coherent structures which are similar to those produced by the proper orthogonal decomposition but which requires only modest statistical input.

1. Introduction

The existence of large-scale, coherent structures in turbulent shear flows is now widely accepted. Although much experimental effort has been directed towards identifying and categorizing such structures (see the reviews by Cantwell 1981 and Robinson 1991 for references) the acquired knowledge of coherent structures has seen only limited direct use in modelling turbulence. This can be attributed, in part, to the lack of a universally agreed upon, unambiguous definition of coherence in a turbulent flow. The Proper Orthogonal Decomposition (POD), first applied to turbulence by Lumley (1967), provides a mathematically rigorous procedure for extracting the most energetic modes from a random field and a means by which the turbulent field can be modelled in terms of coherent structures. Aubry *et al.* (1988) projected the Navier–Stokes equations onto a set of basis functions given by the POD to produce a system of nonlinear ordinary differential equations governing the

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amplitudes of spatial structures in the near-wall region of the turbulent boundary layer. Numerical simulations of this low-order system of equations reproduced the sweeping and bursting phenomena seen experimentally. Recent work along similar lines (cf. Berkooz, Holmes & Lumley 1991; Aubrey & Sanghi 1990; Stone & Holmes 1989) has shown the utility of the POD as a means of reducing complex turbulent dynamics to far simpler systems amenable to analysis by dynamical systems theory. In this way, it now seems that practical active control of turbulence may be possible (Berkooz 1992).

Ideally, one would like to apply the same methodology to a wide range of flows where coherent structures are known to play an important role in the dynamics. The POD procedure, however, requires the two-point velocity autocorrelation tensor as input thus necessitating complete documentation of the flow before the analysis can proceed. For flows with very high Reynolds numbers or complicated geometries this can be prohibitively expensive given current computational and experimental capabilities. The purpose of the present work is to develop an analytic procedure for extracting basis functions (structures) which approximate those given by the POD but which requires much less *a priori* statistical information about the flow.

The method presented is based on energy stability considerations put forth by Lumley (1971). First, the instantaneous flow field is decomposed into three components in order to isolate the large-scale structures. Evolution equations are then written for the coherent velocity field and the coherent kinetic energy. A procedure is formalized to search for the structures which maximize the instantaneous growth rate of coherent energy. The critical assumption is that the structures which on average have the largest growth rate of kinetic energy (the stability modes) will compare well with the structures which contribute the most to the average turbulent kinetic energy (POD eigenfunctions).

The plan of the paper is as follows. The form of the decomposition and the derivation of the evolution equations for the large-scale fields are described in §2. Also, unknowns produced by the decomposition and subsequent averaging are examined. In §3 the maximization procedure is given and the governing eigenvalue relationship is derived. In §4 we propose two closure models and compare results with POD data derived from direct numerical simulations. The role of the mean velocity in the stability procedure is analysed in §5. Here we propose a dynamic model which allows the mean velocity to sense the presence of the large-scale structures. We argue that amplitude behaviour of the simple dynamic model should better compare with the POD eigenspectrum than the eigenspectrum of the stability problem. We show results that support this conjecture and summarize in §6.

2. Derivation of the governing equations

As an example we consider turbulent channel flow assumed statistically homogeneous in both the downstream (x_1) and cross-stream (x_3) directions. In order to extract spatial structures from the total velocity field, we avoid traditional Reynolds averaging and instead decompose the instantaneous field into three components: the spatial mean ($\overline{U}$), the coherent field (v) and the incoherent background turbulence (u'):

$$u_i(\boldsymbol{x}, t) = \overline{U}(x_2) + v_i(\boldsymbol{x}, t) + u_i'(\boldsymbol{x}, t). \tag{2.1}$$

The homogeneous directions allow us to define the mean using spatial averages:

$$\overline{f(\boldsymbol{x},t)} = \frac{1}{L_1 L_3} \int_0^{L_1} \int_0^{L_1} f(\boldsymbol{x},t) \mathrm{d}x_1 \mathrm{d}x_3. \qquad (2.2)$$

We introduce a second averaging procedure, denoted by $\langle \dots \rangle$, which eliminates the small-scale turbulence while leaving the coherent field intact:

$$\langle\, u_i(\boldsymbol{x},t)\,\rangle = \overline{U}(x_2) + v_i(\boldsymbol{x},t). \qquad (2.3)$$

Practically this can be accomplished in several ways. If the structures are assumed to be rollers elongated in the streamwise direction and distributed periodically in x_3 then phase averaging (Hussain & Reynolds 1972; Liu 1988) in this direction is appropriate. For more general configurations spectral estimates such as those proposed by Brereton & Kodal (1992) can be employed. For analytic purposes here, any properly defined, low-pass, spatial filter of the instantaneous field will serve. We require only that the two averages commute,

$$\langle\, \overline{f}\,\rangle = \overline{f}$$

and that the cross-correlations or Leonard stresses are negligible,

$$\langle\, u'v\,\rangle = \overline{u'v} = 0\,.$$

Given these averaging procedures, we can manipulate the Navier–Stokes equations to arrive at evolution equations for the coherent velocity field:

$$\left.\begin{aligned} \mathrm{D}v_i/\mathrm{D}t + v_j v_{i,j} &= -\pi_{,i} + \nu v_{i,jj} + \tau_{ij,j} - U_{i,j} v_j + (\overline{v_i v_j})_{,j}, \\ v_{i,i} &= 0. \end{aligned}\right\} \qquad (2.4)$$

The coherent field depends upon the mean through the convective derivative $\mathrm{D}/\mathrm{D}t = \partial/\partial t + U_j \partial/\partial x_j$ and the mean velocity gradient $U_{i,j}$ and also on the rectified effects of the incoherent small-scale turbulence represented by

$$\tau_{ij} = \overline{u'_i u'_j} - \langle\, u'_i u'_j\,\rangle\,.$$

This quantity can be thought of as a perturbed Reynolds stress which is unknown and will ultimately require modelling. In the limit of a completely random turbulence containing no structure (i.e. $\langle \dots \rangle = 0$) this quantity is equal to the usual Reynolds stress. In the case when the turbulence is completely structured so that $\langle \dots \rangle = \overline{(\dots)}$, τ_{ij} is identically zero. One could write evolution equations for τ_{ij} but they would contain unknown higher-order moments of the unresolved velocities.

The idea of triply decomposing the velocity field to shed light on the nature of large-scale turbulent motions is not new. In a series of papers (Hussain & Reynolds 1970, 1972; Reynolds & Hussain 1972), Hussain and Reynolds introduced controlled wave disturbances into a fully turbulent shear flow. They derived equations for the wave component of the velocity field which are exactly analogous to those given above. These equations were then linearized about a mean turbulent state and the corresponding Orr–Sommerfeld-type relation was solved to find the fastest growing linear wave modes. In this context, the closure problem produced by the decomposition was addressed by using an eddy viscosity argument. Not surprisingly, Reynolds & Tiederman (1967) found that all physically realistic turbulent mean profiles for the channel were stable in the sense of the linear theory. More recently, Butler & Farrell (1992*a*) used non-modal analysis to examine the same linearized problem. By examining the initial value problem thus avoiding the restriction to exponentially growing wave forms, these workers found solutions with large algebraic growth rates indicating a possibility for later secondary, nonlinear instability mechanisms.

In free shear flows, Liu and co-workers have examined both the linear and nonlinear (in the sense of Stuart 1958) behaviour of controlled disturbances in the fully turbulent regime (cf. Liu & Merkine 1976; Alper & Liu 1977; Gatski & Liu 1979; Liu 1988) In this case the (turbulent) mean profile is inflexional so that one can find infinitesimal disturbances with exponential growth rates. Using the 'local linear' theory to determine the shape of the disturbance field, an analytic framework (which we borrow from freely in later sections) for understanding the interactions between the mean flow energy, the coherent field energy and the background turbulence was developed. Extensive predictions of mean spreading rate and control by introduction of upstream disturbances were made.

Our approach here is different in that we do not consider initially infinitesimal perturbations to a turbulent mean velocity profile. Since the reference state is fully turbulent, disturbances of finite amplitude are certainly present. The Reynolds stresses, the sole agency maintaining the mean velocity profile away from the laminar solution, are quadratic in the disturbance velocity. Therefore, realistic analysis of equation (2.4) must account for finite values of the disturbance velocity. To this end we sacrifice the instantaneous information provided by linear stability analysis and turn instead to nonlinear energy methods which allow consideration of perturbations of any size.

3. Energy stability analysis

Energy method analysis has a long history in the context of hydrodynamic stability (cf. Orr 1907; Serin 1959; Joseph 1966). Typically, the method is used to find the characteristic value of some parameter (Re, Ra) below which the flow is absolutely stable, i.e. disturbances of any amplitude decay in time. The equation for the time rate of change of the volume-averaged energy is written. A variational problem can then be defined for the maximum value of the parameter for which the energy growth is zero. Solution of this problem gives the critical value of the parameter and the velocity field corresponding to neutral global stability.

In this work we use the energy method to ask a different question. If we allow for non-zero growth rates, we can find the coherent velocity field satisfying equation (2.4) that at any moment in time has the maximum energy growth rate. This velocity field is the mode that most efficiently extracts energy in a volume-averaged sense from the mean motion while minimizing energy loss to both viscous dissipation and the smaller-scale turbulence. We assume that this maximizing mode will be similar in shape to POD modes which (by definition) contribute the most to the time-averaged turbulent kinetic energy.

This is exactly the idea put forth by Lumley (1971) and investigated by others (cf. Payne 1992) . The present work advances these ideas in several ways. First we give accurate numerical solutions to the resulting eigenvalue problem. By writing the equations in terms of the triple decomposition, the role of the induced stress terms is made clear. We show that the modelling of these terms is crucial and propose a model which gives excellent results. Finally, we are able to compare our results directly with the POD of a full numerical simulation.

As a first step we define the volume-averaged coherent energy, E, and its growth rate λ as a functional of the coherent velocity field:

$$\lambda(v; U_{i,j}, \nu, \tau) = \frac{1}{2E}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\displaystyle\int \left(v_j U_{i,j} - \nu v_{i,jj} + \tau_{ij,j} + \left\{\overline{v_i v_j}_{,j} - v_j v_{i,j} - \pi_{,i}\right\}\right) v_i \mathrm{d}V}{\displaystyle\int v_i v_i \mathrm{d}V}. \quad (3.1)$$

Without making any assumption about the magnitude of the perturbations, the nonlinear pressure and convection terms enclosed in braces can be eliminated using integration by parts, the continuity equation and the boundary conditions on the coherent field. Also, the mean velocity gradient, $U_{i,j}$, appears multiplying the tensor $v_i v_j$ so that only the symmetric part, $S_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$, contributes to the integral. With these manipulations, the growth rate functional becomes

$$\lambda(v; U_{i,j}, \nu, \tau) = \frac{1}{2E}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\displaystyle\int (v_j S_{ij} - \nu v_{i,jj} + \tau_{ij,j}) v_i \mathrm{d}V}{\displaystyle\int v_i v_i \mathrm{d}V}. \quad (3.2)$$

We seek the solenoidal velocity field which maximizes λ. Straightforward application of the calculus of variations gives the Euler equations for the maximizing v field in the form of an eigenvalue relation:

$$\left.\begin{aligned} \lambda v_i + S_{ij} v_j &= -\pi_{,i} + \nu v_{i,jj} + \tau_{ij,j}, \\ v_{i,i} &= 0, \end{aligned}\right\} \quad (3.3)$$

where π now appears as a Lagrange multiplier due to the solenoidal constraint on the velocity field.

Note that equation (3.3) is not the linearized disturbance equation of hydrodynamic stability theory. As shown clearly by Lumley (1971), equation (3.3) and the small disturbance limit of the Navier–Stokes equation cannot be satisfied simultaneously. In the linearized equations, where self-transport of energy is set identically to zero, the energy growth rate is required to be the same at all levels in the flow. Equation (3.3) specifies the global growth rate; locally this quantity may be greater or less, with nonlinear interactions transporting energy spatially as required. While these nonlinear terms are conservative and thus do not enter into the integral expressions used to arrive at the energy stability equation, it is important to note that the disturbances considered are not constrained to be small in any norm. This allows consideration of the stability (in an average sense) of a turbulent flow in which disturbances of finite size not only exist but determine the spatial structure of the mean profile.

We consider coherent fields which are periodic in the homogeneous directions. This allows a decomposition into poloidal and torroidal components which satisfy continuity exactly (Joseph 1973):

$$v_1 = \psi_{,12} - v_{,3} \quad v_2 = -(\psi_{,11} + \psi_{,33}) \quad v_3 = \psi_{,23} + v_{,1}.$$

For velocity fields homogeneous in a single direction, the above decomposition reduces exactly to a velocity/streamfunction formulation. For a channel flow with streamwise invariance, v is simply the streamwise velocity component while ψ is the streamfunction in the (x_2, x_3) plane.

The two scalar functions are then expanded in normal modes in the streamwise and spanwise directions:

$$\psi(\boldsymbol{x}) = \psi(x_2)e^{i(k_1x_1+k_3x_3)},$$
$$v(\boldsymbol{x}) = v(x_2)e^{i(k_1x_1+k_3x_3)}.$$

Substituting the above into equation (3.3) and eliminating the pressure π results in the following coupled two-point differential eigenvalue problem:

$$\left.\begin{aligned} \lambda v + \tfrac{1}{2}ik_3\psi U' &= \nu(D^2-k^2)v - (D\tau_{12} + ik_3\tau_{13}), \\ \lambda(D^2-k^2)\psi + \tfrac{1}{2}ik_3 v U' &= -ik_1(2U'D\psi - U''\psi) + \nu(D^2-k^2)^2\psi \\ &\quad +(D^2+k^2)\tau_{23} + ik_3 D(\tau_{33}-\tau_{22}), \end{aligned}\right\} \quad (3.4)$$

where $D = \partial/\partial x_2$, $k^2 = k_1^2 + k_3^2$.

The boundary conditions on the coherent field are

$$v = D\psi = \psi = 0 \text{ at the walls.}$$

In order to proceed we need to specify a mean velocity profile and a model for the unknown stress terms.

4. Closure models

We have investigated two different closure models for the perturbation stress terms appearing in the eigenvalue relation. The stresses are truly unknowns in that their amplitude and structure have not been determined experimentally. Lacking the physical intuition that comes from experiment, we chose not to use the exact evolution equation for τ_{ij} as a basis for our models. Instead, we resort to a combination of invariant modelling ideas (Lumley 1970) and our knowledge of the unperturbed Reynolds stress.

Modulo the stress terms, equation (3.4) is linear in the coherent velocities. This linearity is an essential advantage of the extraction procedure and for this reason we constrain any stress model to be both linear and homogeneous in the v field and its spatial derivatives. While this seriously limits our choice of models, it ensures that the governing equation remains a regular eigenvalue problem. Further, the evolution equations for the stresses are invariant under uniform translations so the model should depend only on gradients of the coherent velocity. Tensorially this requires

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = \Gamma_{ijkl}(v_{k,l} + v_{l,k}).$$

The nature of the averaging procedure implies that the scales of the coherent field and the background turbulence are different. Assuming that the background turbulence evolves on much shorter time and length scales than the structures, it seems plausible that a Newtonian stress–strain relation like that for the molecular stresses will provide the basis for a model. We set

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = \nu_t(v_{i,j} + v_{j,i}). \quad (4.1)$$

Owing to the inhomogeneity of the turbulence in the wall normal direction, we specify ν_t as a function of x_2 corresponding to experimentally determined values of the traditional eddy viscosity. We will refer to this basic model as the isotropic eddy viscosity model.

Using the basic stress model and an analytic expression for the fully turbulent mean profile (Phillips 1987) we have solved equation (3.4) numerically using Chebyshev

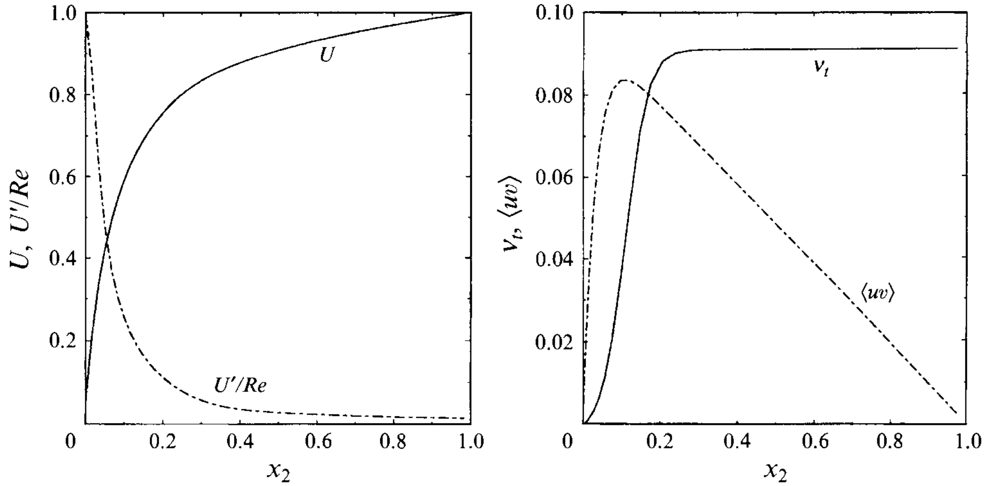

FIGURE 1. Model inputs: (*a*) mean velocity and mean gradient; (*b*) eddy viscosity and Reynolds stress.

expansions and a packaged algorithm for generalized matrix eigenvalue problems. The mean velocity profile and the form of the eddy viscosity are shown in figure 1. Figure 2 shows comparisons between the calculated eigenvectors and the POD results of Moin & Moser (1989) obtained from a numerical channel flow data base. In all figures the wall normal coordinate has been normalized by the channel half-width. Since the equations are invariant under scalar multiplication, we have chosen to normalize the results so that the peak amplitudes of the stability functions and the POD functions are unity.

In general, the stability results indicate that the most unstable modes (those with the largest volume-averaged energy growth rate) are streamwise vorticies in agreement with Butler & Farrell's (1992*b*) 'globally optimal' disturbances, POD analysis and observations of wall-bounded turbulent flows. Reynolds & Tiederman (1967), using the Orr–Sommerfeld equation, found turbulent mean profiles stable for all infinitesimal disturbances; the least-stable disturbance being streamwise propagating waves similar to laminar stability studies. This points again to the fundamental difference between equation (3.3) and the linearized stability equation. The presence of the full rate of strain tensor (S_{ij}) in (3.3) implies that (in the parallel flow case) the vorticity and velocity disturbance equations are cross-coupled, excluding any type of Squire's theorem for reducing the problem to two dimensions. This should be compared with the Orr–Sommerfeld formulation where the normal disturbance vorticity is uncoupled from the streamwise velocity (the so-called Squire mode).

Although there are qualitative similarities in the shape of the stability-based structures and the POD results, the modes predicted by the stability method fall off much more rapidly away from the wall than do the POD functions. The eigenvalue spectra (figure 3) clearly show the stability analysis favouring modes which have a much higher wavelength than the maximum energy modes of the POD.

While there may be a number of reasons for this discrepancy, we choose to first examine more closely the closure model. Since both the POD analysis and the stability method favour modes which are infinitely long in the streamwise direction, we rewrite equation (3.4) using the isotropic eddy viscosity model and setting $k_1 = 0$ for

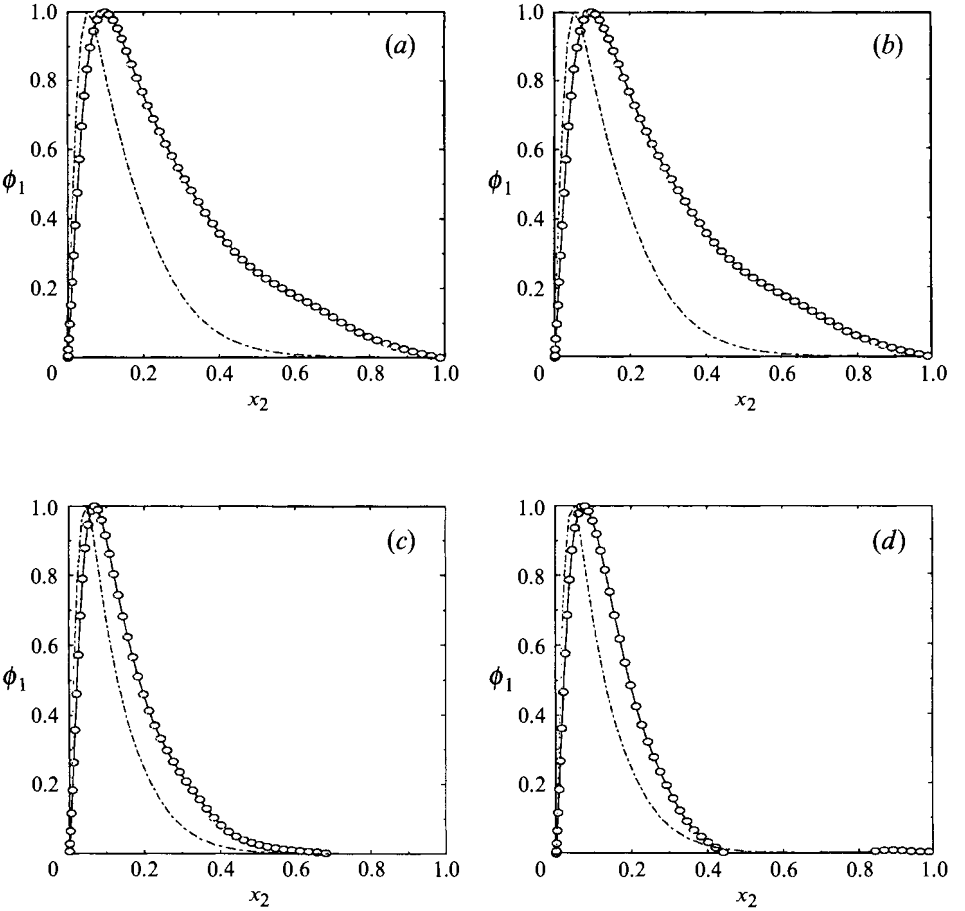

FIGURE 2. First stability eigenfunction ($\phi_1 = v$). Isotropic eddy viscosity model: ○, POD; — - —, isotropic model: (a) $k_3 = 6.00$, (b) $k_3 = 9.00$, (c) $k_3 = 12.00$, (d) $k_3 = 15.00$.

algebraic simplicity:

$$\left.\begin{aligned} \lambda v + \tfrac{1}{2} i k_3 \psi U' &= (v + v_t)(D^2 - k^2)v - v_t' D v, \\ \lambda(D^2 - k^2)\psi + \tfrac{1}{2} i k_3 v U' &= (v + v_t)(D^2 - k^2)^2 \psi \\ &\quad + v_t''(D^2 + k^2)\psi + v_t'(D^2 - k^2) D\psi. \end{aligned}\right\} \quad (4.2)$$

The isotropic eddy viscosity stress model, when applied along with the poloidal velocity decomposition, produces equations for u and ψ which couple only through the action of the mean velocity gradient. For realistic turbulent profiles (see figure 1) regions of appreciable mean shear are confined to thin boundary layers near the wall. Since, in the context of the eigenproblem, the coupling terms act like sources, the eigenfunctions computed using the isotropic model decay away from the wall as quickly as the imposed shear.

From a closure prospective, the isotropic eddy viscosity formulation imposes several severe and possibly unphysical restrictions on the behaviour of the stresses. First, the scalar viscosity ensures that the principle axes of the stress tensor are aligned with the principle axes of the coherent rate of strain or, in terms of the linearized problem considered by Liu (1988), the stresses are phase locked with the rate of strain. When

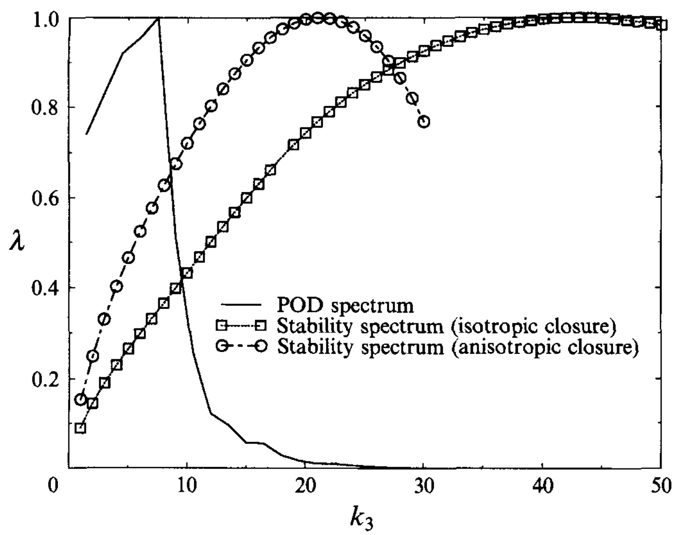

FIGURE 3. Comparison of the POD spectrum with the eigenspectrum of the stability problems

the flow under consideration is independent of one direction (as is the case for the x_1-independent modes in a channel), this implies that the stresses appearing in the equation for v_1 are independent of the gradients of the other velocity components. However, it would appear that the coherent motions in the (x_2,x_3) plane which act to transport fluid across gradients of the mean velocity would contribute significantly to the τ_{1j} stresses.

Secondly, in modelling the perturbed Reynolds stresses by an effective viscosity model, the coherent field plays the role normally occupied by the mean field in typical Reynolds stress closures. Unlike the mean velocity, the coherent field is fully three-dimensional with a large degree of streamline curvature. It can be shown (Pope 1975) that an isotropic viscosity eliminates any dependence of the stress on the mean rotation, limiting the usefulness of such a model in flows with curved streamlines.

We now seek to develop a stress model that allows for some anisotropy in the eddy viscosity and thus couples the component equations through the stress terms. We begin with a standard second-order closure model for the true Reynolds stresses:

$$\frac{\mathrm{D}\overline{u_i u_j}}{\mathrm{D}t} = P_{ij} + \Phi_{ij} - \varepsilon_{ij} \tag{4.3}$$

where $\overline{u_i u_j}$ includes contributions from both the coherent and small-scale turbulence ($u_i = v_i + u_i'$). The pressure–strain correlation is modelled by a return to isotropy term and an isotropization of production term (Naot, Shivit & Wolfshtein 1970),

$$\Phi_{ij} = -\frac{c_1}{T}(\overline{u_i u_j} - \tfrac{1}{3}\overline{u_k u_k}\delta_{ij}) - c_2(P_{ij} - \tfrac{1}{3}P_{kk}\delta_{ij}).$$

The production and dissipation are given by

$$P_{ij} = -\left(\overline{u_i u_k}\frac{\partial \overline{U}_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial \overline{U}_i}{\partial x_k}\right),$$

$$\varepsilon_{ij} = \tfrac{2}{3}\mathscr{E}\delta_{ij}.$$

We assume the stresses are in local equilibrium $D\overline{u_i u_j}/Dt \Rightarrow 0$ and rewrite the model in algebraic form:

$$\frac{c_2}{T}\overline{u_i u_j} = \{(1-c_2)P_{ij} + \tfrac{1}{3}c_2 P_{kk}\delta_{ij}\} - \tfrac{2}{3}(1-c_1)\mathscr{E}\delta_{ij} \tag{4.4}$$

where the turbulent dissipation rate normally denoted by ε is here denoted by $\mathscr{E}$ to avoid confusion with the gauge factor in the perturbation analysis.

Following previous work by Elswick (1967),we set up a perturbation expansion in terms of mean field quantities taking the coherent field as an order-ε perturbation to the spatial mean. We consider the zeroth-order Reynolds stress as known and look for a relationship between the order-ε stress (τ_{ij}) and the order-ε strain rate ($s^1_{ij} = \frac{1}{2}(v_{i,j} + v_{j,k})$) using equation(4.4);

$$\overline{u_i u_j} = R^0_{ij} + \varepsilon R^1_{ij} + \varepsilon^2 \dots ,$$

$$\overline{U}_{i,j} = U^0_{i,j} + \varepsilon U^1_{i,j} + \varepsilon^2 \dots ,$$

$$\mathscr{E} = \mathscr{E}^0 + \varepsilon\mathscr{E}^1 + \varepsilon^2 \dots .$$

Zeroth-order stresses:

$$\left.\begin{aligned} R^0_{23} &= R^0_{32} = 0, \quad R^0_{13} = R^0_{31} = 0, \\ R^0_{33} &= \tfrac{1}{3}R^0_{kk} - \tfrac{2}{3}\mathscr{E}^0 T, \quad R^0_{22} = \tfrac{1}{3}R^0_{kk} - \tfrac{2}{3}\mathscr{E}^0 T, \\ R^0_{11} &= \tfrac{1}{3}R^0_{kk} - \tfrac{2}{3}E^0 T \tfrac{2}{3}\mathscr{E}^0 T - 2U' R^0_{12} T, \\ R^0_{12} &= -U'(R^0_{kk} - \tfrac{2}{3}T\mathscr{E}^0)T. \end{aligned}\right\} \tag{4.5}$$

The order-ε stresses are:

$$\frac{c_2}{T}R^1_{ij} = (1-c_2)P^1_{ij} + \tfrac{1}{3}c_2 P^1_{kk}\delta_{ij} - \tfrac{2}{3}(1-c_1)\mathscr{E}^1\delta_{ij} \tag{4.6}$$

where the production is given by

$$P^1_{ij} = -(R^0_{jk}v_{i,k} + R^0_{ik}v_{j,k} + \dots). \tag{4.7}$$

We have neglected terms containing the mean gradient explicitly for several reasons. First, retaining such terms would lead to the appearance of $U'R^1_{22}$ in the equation for R^1_{12}. Ultimately this term produces a model which is inhomogeneous in the coherent rate of strain thus violating our modelling philosophy. On physical grounds, we argue that the perturbed stress field is due entirely to the presence of the structures and as such we restrict the model to include only production due directly to coherent velocity gradients. This is in agreement with a cascade analogy for the complete flow. The coherent structures are fed energy directly by the mean gradients while the small-scale turbulence is in turn fed by gradients of the coherent field. The appearance of mean production terms at order ε in the perturbation expansion is evidence of leaks in the cascade which we choose to ignore. This does not imply that we have completely neglected the production by the mean velocity, only that we have included it in a homogeneous manner. The off-diagonal terms in the viscosity tensor are exactly the modulated effect of mean production.

If we identify the zeroth-order stresses with an eddy viscosity tensor, then the closure model can be written as

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = -\left(\nu_{ik}v_{j,k} + \nu_{jk}v_{i,k}\right) \tag{4.8}$$

where the tensor viscosity has the following structure in this specific case:

$$\nu_{ij} = \nu_{ij}(R^0) = \begin{pmatrix} \nu_{11} & \nu_{12} & 0 \\ \nu_{12} & \nu_{22} & 0 \\ 0 & 0 & \nu_{22} \end{pmatrix}.$$

The off-diagonal contribution,

$$\nu_{12} = \frac{T}{c_2}(1-c_2)\,R^0_{12}$$

is simply the traditional Reynolds stress which is known experimentally (see figure 1*b*).

Despite the absence of explicit mean production terms, this simple, linear model is still a major improvement over the isotropic eddy viscosity formulation. In the isotropic model the effects of the mean field have been neglected entirely. Here we have allowed for modulation of the perturbation stresses by the mean field through the zeroth-order stresses appearing in the production terms. Also we have unconstrained the model in an important way since the tensorial form of the eddy viscosity allows the principle axes of the stress tensor to be unaligned with the axes of the rate of strain. This is more realistic considering the three-dimensionality of the coherent field.

Applying the model in equation (3.4) leads to

$$\left.\begin{aligned} \lambda v + \tfrac{1}{2}ik_3\psi U' &= (\nu+\nu_{22})(\mathrm{D}^2-k^2)v - \nu_{22}'\mathrm{D}v \\ &\quad + ik_3(\nu_{12}'\mathrm{D}\psi + 2\nu_{12}\mathrm{D}^2\psi), \\ \lambda(\mathrm{D}^2-k^2)\psi + \tfrac{1}{2}ik_3 v U' &= +(\nu+\nu_{22})(\mathrm{D}^2-k^2)^2\psi \\ &\quad + \nu_{22}''(\mathrm{D}^2+k^2)\psi + \nu_{22}'(\mathrm{D}^2-k^2)\mathrm{D}\psi \\ &\quad + ik_3((\nu_{12}''+k_3^2)v + \nu_{12}'v') \end{aligned}\right\} \qquad (4.9)$$

with the expected cross-coupling of the equations through the stress model.

Figure 4 shows eigensolutions to equation (4.9) for several values of k_3. The results compare well with the POD eigenvalues especially for wavenumbers at or below the peak in the POD spectrum. The improvement with decreasing wavenumber is expected given the modelling considerations. The separation of scales between the background turbulence and the coherent structures increases as the wavenumber decreases adding to the expected accuracy of the stress model. The comparison of the two models indicates significant improvements in the results given by the anisotropic eddy viscosity form. The energy method procedure with the more refined closure model appears capable of extracting structures which closely approximate those given by the POD at least at the energy-containing scales of motion.

5. Role of the mean profile

The POD provides a basis set which is ordered in an energy sense. The eigenvalues provide a measure of the kinetic energy in the corresponding eigenfunction. This ordering is an essential advantage of the POD method since it allows one to intelligently select a small number of basis functions with which to represent the motion while ensuring that the energetics of the flow are adequately accounted for. We would like our stability-based extraction method to possess a similar ordering property. From figure 3, it is evident that the eigenspectrum produced by solutions of equation (3.4), while improved by the use of the anisotropic closure model, still predicts structures with maximum growth rate that is a factor of 2 smaller than those containing the

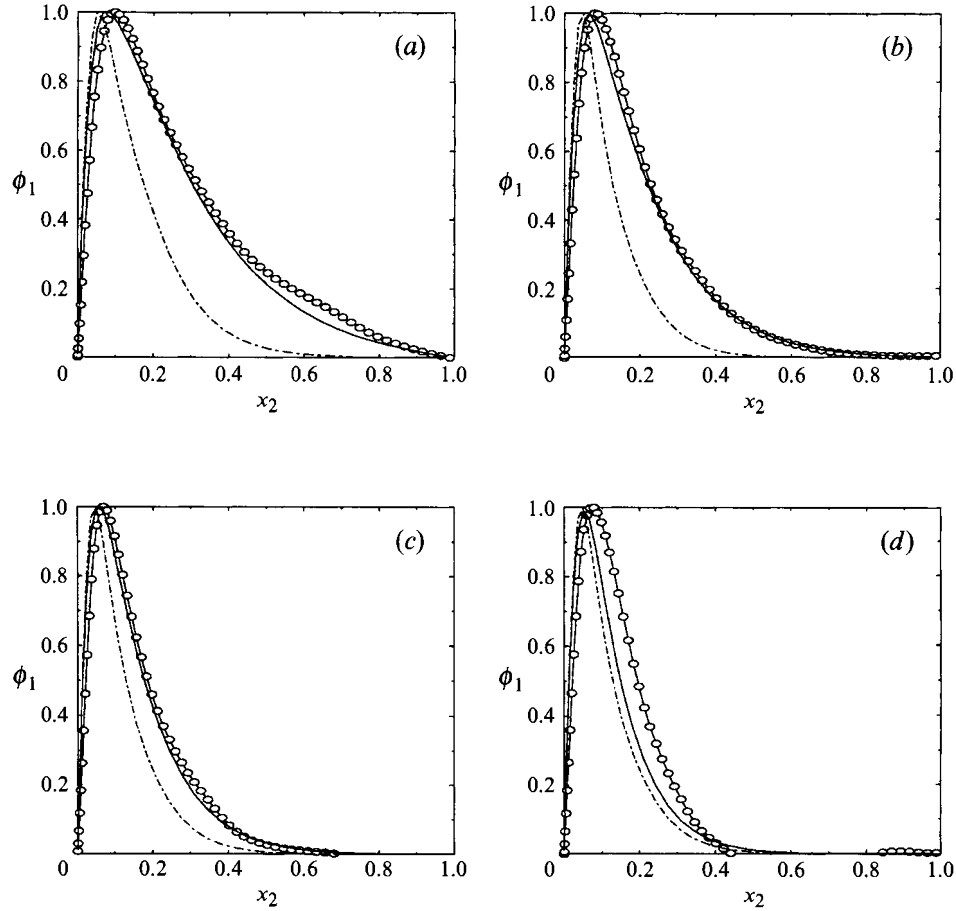

FIGURE 4. First stability eigenfunction ($\phi_1 = v$). Anisotropic eddy viscosity model: *circ*, POD; — - —, isotropic model; —, anisotropic model: (*a*) $k_3 = 6.00$, (*b*) $k_3 = 9.00$, (*c*) $k_3 = 12.00$, (*d*) $k_3 = 15.00$.

most energy as given by the POD. At this point we consider the role of the mean velocity in the two methods.

The POD structures are derived from solutions to the nonlinear Navier–Stokes equations which allow for complicated interaction between the different scales of motion. The structures evolve in a local mean field that is changing in response to the organized motion. Conditionally averaged mean profiles from Blackwelder & Kaplan (1976) clearly show the evolution of the local shear in the presence of coherent structures. In the relatively long period between 'bursting' events, the structures erode the mean shear in which they are embedded.

The stability method on the other hand does not allow for any interaction between the mean and the coherent field. The mean flow is imposed and the resulting structures are calculated. The mean profiles used are time averages which mask any contribution from the coherent field. As such the stability analysis predicts that the highest growth modes are those which can best extract energy from the time-averaged mean shear which is concentrated in the small near-wall region (see figure 1). Since the structures have an aspect ratio close to one, the narrow region of high imposed shear leads to a peak in the eigenspectrum at a large spanwise wavenumber.

We seek a way of relating the POD spectra to the energy stability analysis to arrive at an ordering criterion for the stability eigenfunctions. To do this we follow Liu and co-workers (see Liu 1988 for a complete list of references) and write time evolution equations for the energy density of the coherent field. In this way energy exchanges between the large-scale motions, the small-scale motion and the local mean (modelled with explicit dependence on the coherent velocity) are examined. This is essentially Stuart's (1958) nonlinear stability theory extended to turbulent flow using eigenfunctions derived via the energy method. We expect that the amplitude behaviour of the coherent energy density as a function of cross-stream wavenumber will approximate the average energy content as given by the POD spectrum. At any given time, the true turbulent field is made up of a collection of coherent structures each at a different stage of growth or decay. The time average of the amplitude of a single structure taken over the course of its lifetime is equivalent to the average amplitude of the ensemble of similar structures. By examining the time evolution of a single structure which interacts with the local mean in a physically realistic manner, we can determine the contribution such a structure makes to the ensemble-averaged coherent energy. It seems plausible that the time-averaged energy, as given by the POD spectrum, will be dominated by contributions from structures with the largest ensemble-averaged amplitude.

The exact volume-averaged evolution equations for the coherent and turbulent energies are

$$\frac{\partial \int (v_i v_i)\,\mathrm{d}V}{\partial t} = -\int (v_i v_j)\,\frac{\partial U_i}{\partial x_j}\mathrm{d}V + \int \left(\tau_{ij}\frac{\partial v_i}{\partial x_j}\right)\mathrm{d}V - \nu \int \left(\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\right)\mathrm{d}V, \quad (5.1)$$

$$\frac{\partial \int (u_i u_i)\,\mathrm{d}V}{\partial t} = -\int (u_i u_j)\,\frac{\partial U_i}{\partial x_j}\mathrm{d}V - \int \left(\tau_{ij}\frac{\partial v_i}{\partial x_j}\right)\mathrm{d}V - \int \varepsilon \mathrm{d}V. \quad (5.2)$$

The coherent field is given by the eigenfunctions of the stability problem which now vary in time. Generally, $v_i(\boldsymbol{x},t) = A(t)\phi_i(\boldsymbol{x})$ where the ϕ denote the time-independent eigenfunctions of the stability problem. Specifically,

$$v_1(\boldsymbol{x},t) = A(t)\left(\frac{\partial^2\psi}{\partial x_1 \partial x_2} - \frac{\partial v}{\partial x_3}\right),$$

$$v_2(\boldsymbol{x},t) = -A(t)\left(\frac{\partial^2\psi}{\partial x_1 \partial x_1} + \frac{\partial^2\psi}{\partial x_3 \partial x_3}\right),$$

$$v_3(\boldsymbol{x},t) = A(t)\left(\frac{\partial^2\psi}{\partial x_3 \partial x_2} + \frac{\partial v}{\partial x_1}\right).$$

Similarly, we fix the spatial shape of the background turbulence while allowing its magnitude to adjust with the coherent and mean fields:

$$\int\int u_i u_j \mathrm{d}x_1 \mathrm{d}x_2 = E(t)B_{ij}(x_2). \quad (5.3)$$

The time dependence of the background stress can be inferred from its evolution equation:

$$\frac{\mathrm{D}\tau_{ij}}{\mathrm{D}t} = -\langle u_i u_j \rangle \frac{\partial v_i}{\partial x_j} + \ldots. \quad (5.4)$$

This leads to

$$\tau_{ij}(\boldsymbol{x},t) = A(t)E(t)R_{ij}(\boldsymbol{x}) \tag{5.5}$$

where the anisotropic eddy viscosity model is used to fix the spatial form of the perturbation stresses:

$$R_{ij}(\boldsymbol{x}) = -\left(\nu_{ik}\frac{\partial \phi_j}{\partial x_k} + \nu_{jk}\frac{\partial \phi_i}{\partial x_k}\right). \tag{5.6}$$

For the mean profile we adopt a variation of the 'quasi- steady' model used in Aubrey *et al.* (1988). The equation for the mean profile in a channel can be simplified substantially owing to the homogeneity in the streamwise and cross-stream directions:

$$U(x_2,t) = \frac{1}{\nu}\int_0^{x_2} \langle\, u_1u_2 \,\rangle + v_1v_2\mathrm{d}\hat{x}_2 + \frac{u_\tau^2}{\nu}\left(x_2 - \frac{x_2^2}{2a}\right). \tag{5.7}$$

The background turbulence is assumed to be in local equilibrium with the mean flow and we concentrate on changes in the mean due to the presence of the organized motions. Using this and equation (5.3) the evolving mean profile is given by

$$U(x_2,t) = U(x_2) + \frac{A^2(t)}{\nu}\int_0^{x_2} \phi_1\phi_2\mathrm{d}\hat{x}_2. \tag{5.8}$$

This allows the mean to respond to growing structures providing the necessary feedback to the evolving modes.

Substituting these representations into the energy equations results in a set of coupled ODEs for the temporal evolution of the energies:

$$\left.\begin{aligned} \frac{\mathrm{d}A^2}{\mathrm{d}t} &= A^2\left(I_1(k) - A^2I_2(k) - EI_3\right), \\ \frac{\mathrm{d}E}{\mathrm{d}t} &= A^2EI_3(k). \end{aligned}\right\} \tag{5.9}$$

Again, we have assumed local equilibrium of the background turbulence and the mean velocity by setting mean production of background energy equal to the dissipation. The wavenumber-dependent interaction integrals are given by

$$I_1 = -\int U'\phi_1\phi_2\mathrm{d}V,$$

$$I_2 = \int (\phi_1\phi_2)^2\mathrm{d}V,$$

$$I_3 = \int \nu_{ik}\frac{\partial\phi_j}{\partial x_k}\frac{\partial\phi_i}{\partial x_j} + \nu_{jk}\frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_i}{\partial x_j}\mathrm{d}V\,.$$

Numerical values of the interaction integrals are given in table 1 and the corresponding temporal evolutions of the coherent energy density, A^2, as given by equation (5.9) are shown in figure 5.

Physically, smaller modes given by higher wavenumbers are better able to extract energy from the fixed spatial mean but they lose energy faster to the background turbulence. Also, the smaller modes which have large growth rates initially (as predicted by the stability analysis or the quantity I_2) interact strongly with the mean reducing

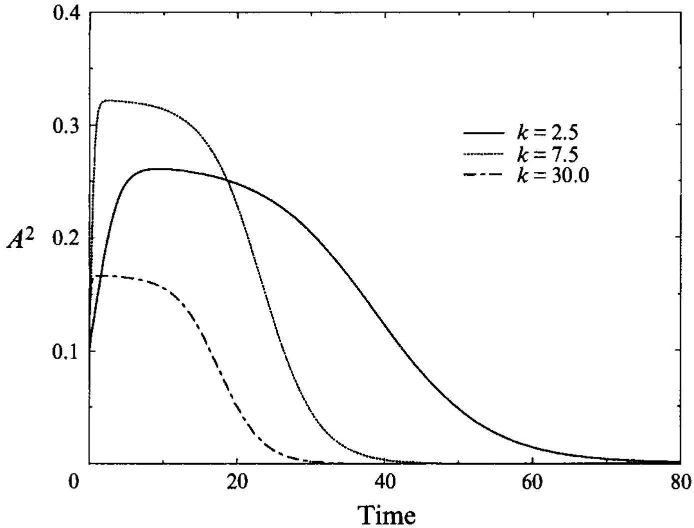

FIGURE 5. Temporal evolution of coherent energy density for parameter values corresponding to different wavenumbers.

k_3	I_1	I_2	I_3
2.5	0.66	2.47	0.508
4.0	2.15	5.33	0.208
5.0	2.99	7.05	0.241
6.0	3.82	8.66	0.425
7.5	4.55	11.0	0.826
10.0	6.76	15.6	1.34
12.5	8.46	20.6	1.71
15.0	10.0	25.6	1.99
20.0	12.8	34.8	2.37
30.0	7.93	47.4	2.05

TABLE 1. Interaction integrals for various wavenumbers

the shear locally. Lower-wavenumber modes are less affected by nonlinear interaction with the mean but grow so slowly that constant energy loss to the background turbulence over time limits their maximum amplitude. Figure 5 indicates that there exists a mode shape corresponding to an intermediate wavenumber which grows to a maximal amplitude before decaying.

In order to quantify comparisons of the single-mode evolution model given by equation (5.9) and the POD eigenspectrum, we define an average of the lifetime of an individual structure. Taking $A^2(t_i) = A^2(t_f)$ where t_i is the initial time and t_f is the final time we designate

$$\{\ldots\} = \frac{1}{T}\int_{t_i}^{t_f} \ldots \mathrm{d}t \qquad (5.10)$$

where $T = t_f - t_i$.

Comparison between $\{A^2\}(k_3)$ and the spectrum of the POD is shown in figure 6. The single-mode model, while not capturing the shape of the POD spectrum exactly, gives a good indication of which wavenumbers are the most energetic. The discrepancy

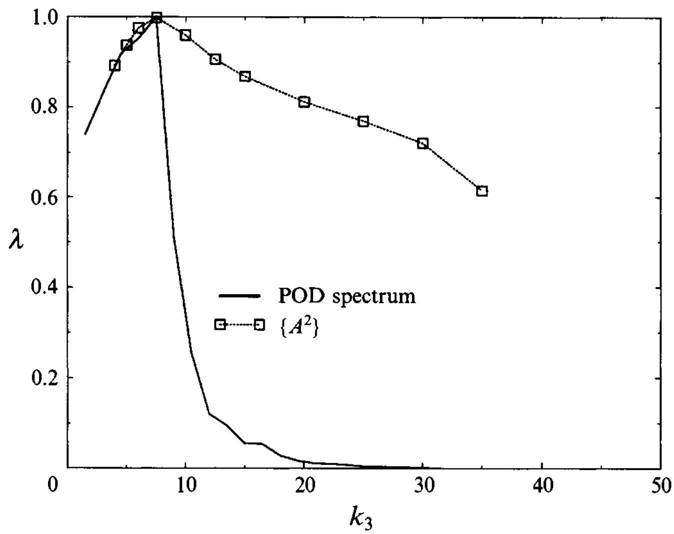

FIGURE 6. Comparison of ensemble-averaged energy content with POD spectrum.

at high wavenumber is not surprising when we consider the simplifications involved in the model. Equation (5.9) describes only the interaction of single mode with the local mean and the background turbulence. The effects of interactions between modes may be negligible for large-scale structures but must become important for smaller scales. Small-scale eddies will see a local shear given not only by the mean field but also by the larger eddies in which they are embedded. Evidence for a dramatic reduction in the ensemble kinetic energy of the smaller scales due to the presence of larger eddies is shown in the Appendix.

6. Conclusions

A method based on nonlinear stability considerations has been presented as a means by which large-scale, energy-containing structures can be extracted from fully turbulent shear flows. After defining two averaging operators and decomposing the instantaneous velocity and pressure fields into the spatial mean, coherent fluctuations and incoherent background turbulence, the evolution equations for the coherent fields were derived. Energy stability analysis of these equations leads to an eigenvalue problem which determines the velocity field that, at any moment in time, maximizes the volume-averaged growth rate of coherent kinetic energy. The form of the maximizing velocity fields was compared with numerical results of the proper orthogonal decomposition (POD) of a turbulent channel flow.

The decomposition and averaging result in equations which are unclosed; the rectified effects of the background fluctuations appear in the form of a perturbed Reynolds stress. A model for these unknowns, based on second-order closure techniques, was derived by perturbation expansion about the mean velocity and Reynolds stress. This led to an anisotropic eddy viscosity which is a function of the unperturbed Reynolds stresses. The eigenfunctions produced by this model compare very well with the POD results for a range of spanwise wavenumbers.

While we cannot guarantee the validity of the stress model in any given flow, preliminary results, using only a simple isotropic viscosity, for the case of sheared convection (Poje 1993) are quite promising. Comparison of the stability-based functions to second-order statistics given by a resolved numerical simulation shows excellent similarity. As is the situation for all known turbulence models, the closure used here for the perturbed Reynolds stress must be considered on a case by case basis for different flow configurations. Such investigations are the subject of future work.

Although the spatial form of POD results were well predicted by the stability structures using the anisotropic closure model, the eigen-spectrum of the stability analysis did not compare well with the POD eigenspectrum. The mode which maximizes the volume-averaged growth rate of coherent energy occurs at a spanwise wavenumber approximately three times larger than that corresponding to the mode containing the largest time-averaged turbulent kinetic energy as predicted by the POD. To explain this discrepancy the role of the mean velocity profile in the two extraction procedures was examined.

The POD procedure is based on solutions of an integral eigenvalue problem where the kernel of the integral operator is the time-averaged, two-point, spatial velocity correlation tensor. Thus, the POD results contain information from the nonlinear Navier–Stokes equations which allow for interaction between the coherent velocity and the mean field. The stability procedure itself cannot account for any dynamic interaction between scales since the mean velocity is imposed at the outset and the resulting structures are then calculated. To examine the physics of interactions between the disparate scales of motion at the same level of complexity as the stability procedure itself, dynamic equations for the evolution of volume-averaged coherent and background turbulent energies were derived. The mean profile was allowed to evolve with the coherent energy based on a model consistent with the mean velocity equation for channel flow.

The coefficients in the evolution equations depend upon integrals of the stability functions and thus on the spanwise wavenumber of the modes. High-wavenumber modes had the largest initial growth as predicted by the eigenvalue of the stability problem. However, the instantaneous growth rate of these modes was reduced quickly due to their strong interaction with the mean profile. The small length scales associated with high wavenumbers lead to fast dissipation to the background field. Large-scale, low-wavenumber modes did not interact strongly with the mean field but had such small initial growth rates that energy loss to the background turbulence inhibited the maximum-amplitude achieved. It was found that modes corresponding to intermediate wavenumbers ($k_3 \approx 7.5$) grow to the largest amplitude. These maximum amplitude structures compare very well with the POD eigenfunctions containing the most time-averaged kinetic energy.

In order to study the dynamics of coherent structures which exist in a turbulent flow we require a means by which the organized part of the flow can be extracted from the random fluctuations. The energy-stability-based procedure proposed here provides an extraction method that stems from the Navier–Stokes equations. The results compare well with those given by the proper orthogonal decomposition but require substantially less *a priori* statistical information about the flow. By reducing the initial experimental or numerical costs we hope to use this procedure as first step in examining the dynamics of coherent structures in flows that until now have not been amenable to the proper orthogonal decomposition.

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Appendix

In order to estimate the effects of larger-scale motions on the evolution of the smaller scales, we examine the interaction between a fundamental wave disturbance (small scale) and its subharmonic (large scale). The coherent field is then made up of two components, $v_i = \tilde{v}_i + \hat{v}_i$ where the fundamental, $\hat{v}$, is periodic in x_3 with wavelength $\lambda/2$ and the subharmonic, $\tilde{v}$, has wavelength λ.

Since the periods of the disturbances have been artificially prescribed, two phase averages can be defined to decompose the velocity field analytically:

$$\langle f \rangle = \frac{1}{N}\sum_n f(x_3 + n\lambda), \tag{A 1}$$

$$\langle\langle f \rangle\rangle = \frac{1}{N}\sum_n f(x_3 + \tfrac{1}{2}n\lambda). \tag{A 2}$$

In this way, the coherent contribution from the instantaneous velocity is given by

$$\langle \tilde{U} \rangle = \overline{U} + \hat{v} + \tilde{v},$$

and the fundamental can be separated from the subharmonic using the $\langle\langle\,\rangle\rangle$ average:

$$\langle\langle \hat{v} + \tilde{v} \rangle\rangle = \hat{v}.$$

The volume-averaged kinetic energy equations for the two components of the coherent field are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\tilde{v}_i\tilde{v}_i)\,\mathrm{d}V = -\int\left(\tilde{v}_i\tilde{v}_j\frac{\partial \overline{U}_i}{\partial x_j} + \tilde{v}_i\tilde{v}_i\frac{\partial \hat{v}_i}{\partial x_j}\right)\mathrm{d}V + \ldots, \tag{A 3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\hat{v}_i\hat{v}_i)\,\mathrm{d}V = -\int\left(\hat{v}_i\hat{v}_j\frac{\partial \overline{U}_i}{\partial x_j} - \tilde{v}_i\tilde{v}_i\frac{\partial \hat{v}_i}{\partial x_j}\right)\mathrm{d}V + \ldots. \tag{A 4}$$

As before, the coherent velocities are taken as the eigenfunctions of the stability problem,

$$\hat{v}_i = A_1(t)\phi_i^1(x), \quad \tilde{v}_i = A_1(t)\phi_i^2(x)$$

and the mean velocity model now contains contributions from both the fundamental and the subharmonic,

$$\frac{\partial \overline{U}}{\partial x_2} = U' + (\hat{v}_1 + \tilde{v}_1)(\hat{v}_2 + \tilde{v}_2).$$

New coupling terms appear in the evolution equations for the coherent energy densities due to interaction between the different size modes:

$$\left.\begin{array}{rcl} \partial A_1^2/\partial t & = & A_1^2\left(I_1^1 - I_2^{11}A_1^2 - I_2^{12}A_2^2 - EI_3^1\right) - A_1A_2^2I_4, \\ \partial A_2^2/\partial t & = & A_2^2\left(I_1^2 - I_1^{21}A_2^2 - I_2^{22}A_2^2 - EI_3^2\right) + A_1A_2^2I_4, \\ \partial E/\partial t & = & E\left(A_1^2I_3^1 + A_2^2I_3^2\right). \end{array}\right\} \tag{A 5}$$

I_2^{12} and I_2^{21} represent the effect on the mean shear production of fundamental (subharmonic) coherent energy due to the presence of the subharmonic (fundamental). I_4 is a measure of the direct energy transfer between the two modes due to the working of the subharmonic stresses against the fundamental rate of strain. This quantity appears with opposite sign in each equation.

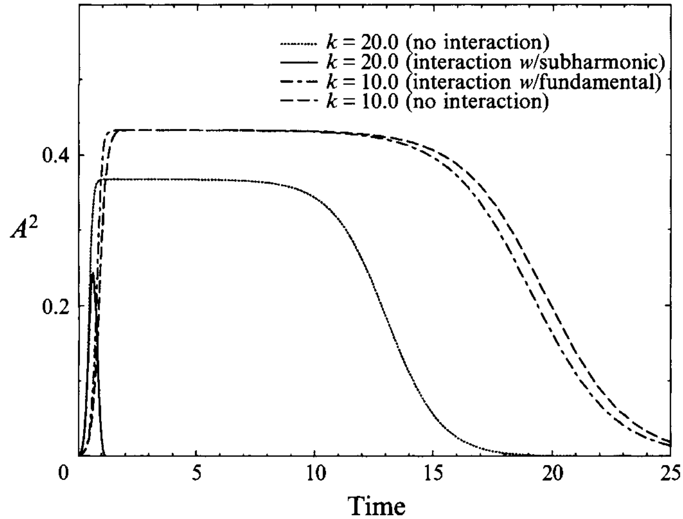

FIGURE 7. Temporal evolution of coherent energy densities showing the effect of mode–mode interaction.

As an example, we have calculated the the interaction integrals using the mode $k_3 = 20.0$ as the fundamental. For this case,

$$I_1 = (12.8, 676)\,,\ I_2 = \begin{pmatrix} 34.8 & 47.1 \\ 91.8 & 15.6 \end{pmatrix},\ I_3 = (2.37, 1.34)\,,\ I_4 = 2.8.$$

The results of integrating equation (A 5) for these coefficient values are shown in figure 7. The mode–mode dynamics are dominated by terms due to the mean velocity feedback model. The direct interaction term, I_4, is much smaller. The effect of the large-scale subharmonic on the behaviour of the small-scale fundamental is dramatic. Although I_2^{21} is twice as large as I_2^{12}, the slowly growing subharmonic mode is practically unaffected by the presence of the fundamental. The fundamental, however, is quickly damped by the larger-scale motion. This agrees with our intuitive picture of the physics. The small-scale motions see not only the mean shear but the strain rate due to all larger scales.

While the situation analyzed here is admittedly artificial since a real turbulent flow will contain structures of all sizes at any given instant, there is no reason not to expect a qualitatively similar effect on the smaller-scale motions in the real flow. Since the larger eddies are relatively unaffected by the smaller scales, the arguments presented in §5 and the small-wavenumber results shown in figure 6 will carry over to the case of many interacting eddies. The spuriously slow fall off in the spectrum of $\{A^2\}$, however, would be eliminated.

REFERENCES

ALPER, A. & LIU, J. T. C. 1977 On the interaction between large scale structure and fine-grained turbulence in a free shear flow. II. The development of spatial interactions in the mean. *Proc. R. Soc. Lond.* A **359**, 497–523.

AUBRY, N., HOLMES, P., LUMLEY, J. L. & STONE, E. 1988 The dynamics of coherent structures in the wall region of a turbulent boundary layer. *J. Fluid Mech.* **192**, 115–197.

AUBREY, N. & SANGHI, S. 1990 Bifurcation and bursting of streaks in the turbulent wall layer. In *Organized Structures and Turbulence in Fluid Mechanics* (ed. M. Lesieur and O. Métais). Kluwer.

BERKOOZ, G. 1992 Feedback control by boundary deformation of models of the turbulent wall layer. *Tech. Rep.* FDA-92-16. Sibley School of Mechanical and Aerospace Engineering, Cornell University.

BERKOOZ, G., HOLMES, P. & LUMLEY, J. L. 1991 Intermittent dynamics in simple models of the turbulent wall layer. *J. Fluid Mech.* **230**, 75–95.

BLACKWELDER, R. F. & KAPLAN, R. E. 1976 On the wall structure of the turbulent boundary layers. *J. Fluid Mech.* **76**, 89–112.

BRERETON, G. J. & KODAL, A. 1992 A frequency-domain filtering technique for triple decomposition of unsteady turbulent flow. *J. Fluids Engng* **114**, 45–51.

BUTLER, K. M. & FARRELL, B. F. 1992*a* Optimal perturbations in wall-bounded turbulent shear flow. *Bull. Am. Phys. Soc.* **37**, 1738.

BUTLER, K. M. & FARRELL, B. F. 1992*b* Three-dimensional optimal perturbations in viscous shear flow. *Phys. Fluids* A, **4**, 1637–1650.

CANTWELL, B. J. 1981 Organized motions in turbulent flow. *Ann. Rev. Fluid Mech.* **13**, 457–515.

ELSWICK, R. C. 1967 Investigation of a theory for the structure of the viscous sublayer in wall turbulence. Master's thesis, The Pennsylvania State University.

GATSKI, T. B. & LIU, J. T. C. 1979 On the interaction between large scale structure and fine-grained turbulence in a free shear flow. III. A numerical solution. *Phil. Trans. R. Soc. Lond.* A **293**, 473–509.

HUSSAIN, A. K. M. F. & REYNOLDS, W. C. 1970 The mechanics of an organized wave in turbulent shear flow. *J. Fluid Mech.* **41**, 241–258.

HUSSAIN, A. K. M. F. & REYNOLDS, W. C. 1972 The mechanics of an organized wave in turbulent shear flow. Part 2. Experimental results. *J. Fluid Mech.* **54**, 241–262.

JOSEPH, D. D. 1966 Nonlinear stability of the Boussinesq equations by the method of energy. *Arc. for Rat. Mech. Anal.* **22**, 163.

JOSEPH, D. D. 1973 *Stability of Fluid Flow*. Springer.

LIU, J. T. C. 1988 Contributions to the understanding of large-scale coherent structures in developing free turbulent shear flows. *Adv. Appl. Mech.* **26**, 183–309.

LIU, J. T. C. & MERKINE, L. 1976 On the interaction between large scale structure and fine-grained turbulence in a free shear flow. I. The development of temporal interactions in the mean. *Proc. R. Soc. Lond.* A **352**, 213–247.

LUMLEY, J. L. 1967 The structure of inhomgeneous turbulence. In *Atmospheric Turbulence and Tave Propagation*. Nauka.

LUMLEY, J. L. 1970 Towards a turbulent constitutive relation. *J. Fluid Mech.*, **41**, 413–434.

LUMLEY, J. L. 1971 Some comments on the energy method. In *Developments in Mechanics 6* (ed. L. Lee & A. Szewczyk). Notre Dame Press.

MOIN, P. & MOSER, R. 1989 Characteristic eddy decomposition of turbulence in a channel. *J. Fluid Mech.* **200**, 471–500.

NAOT, D., SHIVIT, A. & WOLFSHTEIN, M. 1970 Interactions between componenents of the turbulent velocity correlation tensor. *Israel J. Tech.* **8**, 259.

ORR, W. M. 1907 The stability or instability of steady motions in a liquid. Part II: A viscous liquid. *Proc. R. Irish Acad.* A **28**, 122.

PAYNE, F. R. 1992 Lumley's PODT definition of large eddies and a trio of numerical procedures. In *Studies in Turbulence* (ed. T. B. Gatski, S. Sarkar & C. G. Speziale). Springer-Verlag.

PHILLIPS, W. R. C. 1987 The wall region of a turbulent boundary layer. *Phys. Fluids* **30**, 2354–2361.

POJE, A. C. 1993 An energy method stability model for large scale structures in turbulent shear flows. PhD thesis, Cornell University.

POPE, S. B. 1975 A more general effective–viscosity hypothesis. *J. Fluid Mech.* **72**, 331–340.

REYNOLDS, W. C. & HUSSAIN, A. K. M. F. 1972 The mechanics of an organized wave in turbulent shear flow. Part 3. Theoretical models and comparisons with experiment. *J. Fluid Mech.* **54**, 263–287.

Reynolds, W. C. & Tiederman, W. G. 1967 Stability of turbulent channel flow with application to Malkus's theory. *J. Fluid Mech.* **27**, 253–272.

Robinson, S. K. 1991 Coherent motions in the turbulent boundary layer. *Ann. Rev. Fluid Mech.* **23**, 601–639.

Serrin, J. 1959 On the stability of viscous fluid motions. *Arch. Rat. Mech. Anal.* **3**, 1.

Stone, E. & Holmes, P. 1989 Noise induced intermittency in a model of a turbulent boundary layer. *Physica* **37D**, 20–32.

Stuart, J. T. 1958 On the non-linear mechanics of hydrodynamic stability. *J. Fluid Mech.* **4**, 1–21.